

A random regularized approximate solution of the inverse problem for the Burgers' equation

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Abstract

In this paper, we find a regularized approximate solution for an inverse problem for the Burgers' equation. The solution of the inverse problem for the Burgers' equation is ill-posed, i.e., the solution does not depend continuously on the data. The approximate solution is the solution of a regularized equation with randomly perturbed coefficients and randomly perturbed final value and source functions. To find the regularized solution, we use the modified quasi-reversibility method associated with the truncated expansion method with nonparametric regression. We also investigate the convergence rate.

1 Introduction

In this work, we consider the backward in time problem for **1-D Burgers' equation**

$$\begin{cases} \mathbf{u}_t - (A(x, t)\mathbf{u}_x)_x &= \mathbf{u}\mathbf{u}_x + G(x, t), & (x, t) \in \Omega \times (0, T), \\ \mathbf{u}(x, t) &= 0, & x \in \partial\Omega, \\ \mathbf{u}(x, T) &= H(x), & x \in \Omega, \end{cases} \quad (1.1)$$

where $\Omega = (0, \pi)$. The Burgers equation is a fundamental partial differential equation occurring in various areas of applied mathematics, such as fluid mechanics, nonlinear acoustics, gas dynamics, traffic flow [7].

One can see that the term $(A(x, t)\mathbf{u}_x)_x$ is $\Delta\mathbf{u} = \mathbf{u}_{xx}$ if $A = 1$. However, one can not use spectral methods to study the operator $(A(x, t)\mathbf{u}_x)_x$. So, the problem is more difficult. The second observation is that for the equation $\mathbf{u}_t - (A(x, t)\mathbf{u}_x)_x = f(\mathbf{u}, \mathbf{u}_x)$ when $A(x, t)$ is deterministic and $f(\mathbf{u}, \mathbf{u}_x) = f(\mathbf{u})$, the problem is a consequence of Theorem 4.1 in our recent paper [4]. However, if $A(x, t)$ is randomly perturbed and $f(\mathbf{u}, \mathbf{u}_x)$ depends on \mathbf{u} and \mathbf{u}_x then the problem is more challenging.

Until now, the deterministic Burgers' equation with the randomly perturbed case have not been studied. Hence, the paper is the first study of Burgers' equation backward in time. The inclusion of the gradient term in uu_x in the right hand side of the Burgers' equation

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makes the Burgers' equation more difficult to study. We need to find an approximate function for $\mathbf{u}\mathbf{u}_x$. This task is nontrivial.

This paper is a continuation of our study of backward problems in the two recent papers [4, 5]. In those papers the equations did not have random coefficients in the main equations. The paper [4] does not consider the random operator. The paper [5] considers the simple coefficient $A(x, t) = A(t)$ and the source function is $\mathbf{u} - \mathbf{u}^3$. Hence, one can see that the Burgers' equation considered here is more difficult since the gradient term in the right hand side and the coefficient $A(x, t)$ depends on both x and t .

It is known that the backward problem mentioned above is ill-posed in general [7], i.e., a solution does not always exist. When the solution exists, the solution does not depend continuously on the given initial data. In fact, from a small noise of a physical measurement, the corresponding solution may have a large error. This makes the numerical computation troublesome. Hence a regularization is required. It is well-known that there are some difficulties to study the nonlocal Burger's equation. First, by the given form of coefficient $A(x, t)$ in the main equation (1.1), the solution of Problem (1.1) can not be transformed into a nonlinear integral equation. Hence, classical spectral method cannot be applied. The second thing that makes the Burger's equation more difficult to study is the gradient term \mathbf{u}_x in the right hand side. Until now, although there are limited number of works on the backward problem for Burgers' equation [1, 3], there are no results for regularizing the problem.

As is well-known, measurements always are given at a discrete set of points and contain errors. These errors may be generated from controllable sources or uncontrollable sources. In the first case, the error is often deterministic. If the errors are generated from uncontrollable sources as wind, rain, humidity, etc., then the model is random. Methods used for the deterministic cases cannot be applied directly to the random case.

In this paper, we consider the following model as follows

$$\tilde{H}(x_k) = H(x_k) + \sigma_k \epsilon_k, \quad \tilde{G}_k(t) = G(x_k, t) + \vartheta \xi_k(t), \quad \text{for } k = \overline{1, n}, \quad (1.2)$$

and

$$\tilde{A}_k(t) = A(x_k, t) + \bar{\vartheta} \xi_k(t), \quad \text{for } k = \overline{1, n}. \quad (1.3)$$

where $x_k = \pi \frac{2k-1}{2n}$ and ϵ_k are unknown independent random errors. Moreover, $\epsilon_k \sim \mathcal{N}(0, 1)$, and $\sigma_k, \vartheta, \bar{\vartheta}$ are unknown positive constants which are bounded by a positive constant V_{\max} , i.e., $0 \leq \sigma_k < V_{\max}$ for all $k = 1, \dots, n$. $\xi_k(t)$ are Brownian motions. The noises $\epsilon_k, \xi_k(t)$ are mutually independent. Our task is reconstructing the initial data $u(x, 0)$.

We next want to mention about the organization of the paper and our methods in this paper. We prove some preliminary results in section 2. We state and prove our main result in section 3. The existence and uniqueness of solution of equation (1.1) is an open problem, and we do not investigate this problem here. For inverse problem, we assume that the solution of the Burgers' equation (1.1) exists. In this case its solution is not stable. In this paper we establish an approximation of the backward in time problem for 1-D Burgers' equation (1.1) with the solution of a regularized equation with randomly perturbed equation (2.13). The random perturbation in equation (2.13) is explained in equations (1.2), (1.3), (2.13) and (2.14).

2 Some Notation

We first introduce notation, and then state the first set of our main results in this paper. We define fractional powers of the Neumann-Laplacian.

$$Af(x) := -\Delta f(x) = -\frac{\partial^2 f(x)}{\partial x^2}. \quad (2.4)$$

Since A is a linear densely defined self-adjoint and positive definite elliptic operator on the connected bounded domain $\Omega = (0, \pi)$ with Dirichlet boundary condition, the eigenvalues of A satisfy

$$\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_p \leq \dots$$

with $\lambda_p = p^2 \rightarrow \infty$ as $p \rightarrow \infty$. The corresponding eigenfunctions are denoted respectively by $\varphi_p(x) = \sqrt{\frac{2}{\pi}} \sin(px)$. Thus the eigenpairs (λ_p, ϕ_p) , $p = 0, 1, 2, \dots$, satisfy

$$\begin{cases} A\varphi_p(x) = -\lambda_p\phi_p(x), & x \in \Omega \\ \partial_x \phi_p(x) = 0, & x \in \partial\Omega. \end{cases}$$

The functions φ_p are normalized so that $\{\phi_p\}_{p=0}^\infty$ is an orthonormal basis of $L^2(\Omega)$.

Defining

$$H^\gamma(\Omega) = \left\{ v \in L^2(\Omega) : \sum_{p=0}^\infty \lambda_p^{2\gamma} |\langle v, \phi_p \rangle|^2 < +\infty \right\},$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(\Omega)$, then $H^\gamma(\Omega)$ is a Hilbert space equipped with norm

$$\|v\|_{H^\gamma(\Omega)} = \left(\sum_{p=1}^\infty \lambda_p^{2\gamma} |\langle v, \phi_p \rangle|^2 \right)^{1/2}.$$

First, we state following Lemmas that will be used in this paper

Theorem 2.1 (Theorem 2.1 in [4]). *Define the set \mathcal{W}_{β_n} for any $n \in \mathbb{N}$*

$$\mathcal{W}_{\beta_n} = \left\{ p \in \mathbb{N} : |p| \leq \sqrt{\beta_n} \right\} \quad (2.5)$$

where β_n satisfies

$$\lim_{|n| \rightarrow +\infty} \beta_n = +\infty.$$

For a given n and β_n we define functions that are approximating H, G as follows

$$\hat{H}_{\beta_n}(x) = \sum_{p \in \mathcal{W}_{\beta_n}} \left[\frac{\pi}{n} \sum_{k=1}^n \tilde{D}_k \psi_p(x_k) \right] \psi_p(x), \quad \hat{G}_{\beta_n}(x, t) = \sum_{p \in \mathcal{W}_{\beta_n}} \left[\frac{\pi}{n} \sum_{k=1}^n \tilde{G}_k(t) \psi_p(x_k) \right] \psi_p(x). \quad (2.6)$$

Let us choose $\mu_0 > \frac{1}{2}$. If $H \in \mathcal{H}^{\mu_0}(\Omega)$ and $G \in L^\infty(0, T; \mathcal{H}^{\mu_0}(\Omega))$ then the following estimates hold

$$\begin{aligned} \mathbf{E} \left\| \hat{H}_{\beta_n} - H \right\|_{L^2(\Omega)}^2 &\leq \overline{C}(\mu_0, H) \sqrt{\beta_n} n^{-4\mu_0} + 4\beta_n^{-\mu_0} \left\| H \right\|_{\mathcal{H}^{\mu_0}(\Omega)}^2, \\ \mathbf{E} \left\| \hat{G}_{\beta_n}(\cdot, t) - G(\cdot, t) \right\|_{L^\infty(0, T; L^2(\Omega))}^2 &\leq \overline{C}(\mu_0, G) \sqrt{\beta_n} n^{-4\mu} + 4\beta_n^{-\mu_0} \left\| G \right\|_{L^\infty(0, T; \mathcal{H}^{\mu_0}(\Omega))}^2, \end{aligned}$$

where

$$\overline{C}(\mu_0, H) = 8\pi V_{max}^2 \frac{2\pi^{1/2}}{\Gamma(1/2)} + \frac{16C^2\mu_0\pi^{1/2}}{\Gamma(1/2)} \|H\|_{\mathcal{H}^{\mu_0}(\Omega)}^2.$$

and

$$\overline{C}(\mu_0, G) = 8\pi V_{max}^2 \frac{2\pi^{1/2}}{\Gamma(1/2)} + \frac{16C^2\mu\pi^{1/2}}{\Gamma(1/2)} \|G\|_{L^\infty(0,T;\mathcal{H}^{\mu_0}(\Omega))}^2.$$

Corollary 2.1 (Corollary 2.1 in [4]). *Let H, G be as in Theorem (2.1). Then the term $\mathbf{E} \|\widehat{H}_{\beta_n} - H\|_{L^2(\Omega)}^2 + T\mathbf{E} \|\widehat{G}_{\beta_n} - G\|_{L^\infty(0,T;L^2(\Omega))}^2$ is of order*

$$\max \left(\sqrt{\beta_n} n^{-4\mu_0}, \beta_n^{-\mu_0} \right).$$

Lemma 2.1. *Define the following space of functions*

$$\mathcal{Z}_{\gamma,B}(\Omega) := \left\{ f \in L^2(\Omega), \sum_{p \in \mathbb{N}} p^{2+2\gamma} e^{2Bp^2} \langle f, \psi_p \rangle_{L^2(\Omega)}^2 < +\infty \right\}, \quad (2.7)$$

for any $\gamma \geq 0$ and $B \geq 0$. Define also the operator $\mathbf{P} = A_1 \Delta$ and \mathbf{P}_{ρ_n} is defined as follows

$$\mathbf{P}_{\rho_n}(v) = A_1 \sum_{p \leq \sqrt{\frac{\rho_n}{A_1}}} p^2 \langle v(x), \psi_p \rangle_{L^2(\Omega)} \psi_p, \quad (2.8)$$

for any function $v \in L^2(\Omega)$. Then for any $v \in L^2(\Omega)$

$$\|\mathbf{P}_{\rho_n}(v)\|_{L^2(\Omega)} \leq \rho_n \|v\|_{L^2(\Omega)}, \quad (2.9)$$

and for $v \in \mathcal{Z}_{\gamma,TA_1}(\Omega)$ then

$$\|\mathbf{P}v - \mathbf{P}_{\rho_n}v\|_{L^2(\Omega)} \leq A_1 \rho_n^{-\gamma} e^{-T\rho_n} \|v\|_{\mathcal{Z}_{\gamma,TA_1}(\Omega)}. \quad (2.10)$$

Proof. First, for any $v \in L^2(\Omega)$, we have

$$\begin{aligned} \|\mathbf{P}_{\rho_n}(v)\|_{L^2(\Omega)}^2 &= A_1^2 \sum_{p \leq \sqrt{\frac{\rho_n}{A_1}}} p^4 \langle v(x), \psi_p \rangle_{L^2(\Omega)}^2 \\ &\leq \rho_n^2 \sum_{p \leq \sqrt{\frac{\rho_n}{A_1}}} \langle v(x), \psi_p \rangle_{L^2(\Omega)}^2 = \rho_n^2 \|v\|_{L^2(\Omega)}^2, \end{aligned} \quad (2.11)$$

and

$$\begin{aligned} \|\mathbf{P}v - \mathbf{P}_{\rho_n}(v)\|_{L^2(\Omega)}^2 &= A_1^2 \sum_{p > \sqrt{\frac{\rho_n}{A_1}}} p^{-4\gamma} e^{-2TA_1|p|^2} p^{4+4\gamma} e^{2TA_1p^2} \langle v(x), \psi_p \rangle_{L^2(\Omega)}^2 \\ &\leq A_1^2 \rho_n^{-2\gamma} e^{-2TA_1\rho_n} \sum_{p > \sqrt{\frac{\rho_n}{A_1}}} p^{4+4\gamma} e^{2TA_1p^2} \langle v(x), \psi_p \rangle_{L^2(\Omega)}^2 \\ &= A_1^2 \rho_n^{-2\gamma} e^{-2T\rho_n} \|v\|_{\mathcal{Z}_{\gamma,TA_1}(\Omega)}^2. \end{aligned} \quad (2.12)$$

□

Now, we can assume that $\hat{A}_{\rho_n}(x, t), A(x, t) \leq A_0$ for all $(x, t) \in \Omega \times (0, T)$ and we choose $A_1 > A_0$. We describe our regularized problem by defining the following problem

$$\left\{ \begin{array}{l} \frac{\partial \tilde{U}_{\rho_n, \beta_n}}{\partial t} - \left(\hat{A}_{\beta_n}(x, t) \frac{\partial \tilde{U}_{\rho_n, \beta_n}}{\partial x} \right)_x - \mathbf{P} \tilde{U}_{\rho_n, \beta_n} + \mathbf{P}_{\rho_n} \tilde{U}_{\rho_n, \beta_n} = \\ \quad = \bar{\mathbf{F}}^0_{\hat{Q}_n} \left(\tilde{U}_{\rho_n, \beta_n}, \frac{\partial \tilde{U}_{\rho_n, \beta_n}}{\partial x} \right) + \hat{G}_{\rho_n}(x, t), \quad 0 < t < T, \\ \tilde{U}_{\rho_n, \beta_n}(x, t) = 0, \quad x \in \partial\Omega, \\ \tilde{U}_{\rho_n, \beta_n}(x, T) = \hat{H}_{\beta_n}(x). \end{array} \right. \quad (2.13)$$

Here \hat{A}_{β_n} is defined by

$$\hat{A}_{\beta_n}(x, t) = \sum_{p \in \mathcal{W}_{\beta_n}} \left[\frac{\pi}{n} \sum_{k=1}^n \tilde{A}_k(t) \psi_p(x_k) \right] \psi_p(x) \quad (2.14)$$

where $\psi_p(x) = \sqrt{\frac{2}{\pi}} \sin(px)$. Noting as above, the function $F(\mathbf{u}, \mathbf{u}_x) = \mathbf{u} \mathbf{u}_x$ in the first equation of Problem (1.1) is locally Lipschitz function and is approximated by the function $\bar{\mathbf{F}}^0_{\hat{Q}_n} \left(\tilde{U}_{\rho_n}, \frac{\partial \tilde{U}_{\rho_n}}{\partial x} \right)$ in the first equation of Problem (2.13) where

$$\bar{\mathbf{F}}^0_{\hat{Q}_n}(v, \hat{v}) := \begin{cases} \hat{Q}_n^2, & \max\{v, \hat{v}\} \in (\hat{Q}_n, +\infty), \\ v\hat{v}, & \max\{v, \hat{v}\} \in [-\hat{Q}_n, \hat{Q}_n], \\ \hat{Q}_n^2, & \max\{v, \hat{v}\} \in (-\infty, -\hat{Q}_n). \end{cases} \quad (2.15)$$

Here the function \hat{Q}_n is increasing function and $\lim_{n \rightarrow +\infty} \hat{Q}_n = +\infty$. For a sufficiently large $n > 0$ such that

$$\hat{Q}_n \geq \max \left(\|\mathbf{u}\|_{L^\infty((0, T); L^2(\Omega))}, \|\mathbf{u}_x\|_{L^\infty((0, T); L^2(\Omega))} \right).$$

We show that $\bar{\mathbf{F}}^0_{\hat{Q}_n}$ is a globally Lipschitz function by the following Lemma

Lemma 2.2. *For any $(v, \hat{v}) \in \mathbb{R}^2, (w, \hat{w}) \in \mathbb{R}^2$, we obtain*

$$\left| \bar{\mathbf{F}}^0_{\hat{Q}_n}(v, \hat{v}) - \bar{\mathbf{F}}^0_{\hat{Q}_n}(w, \hat{w}) \right| \leq \hat{Q}_n (|v - \hat{v}| + |w - \hat{w}|). \quad (2.16)$$

Proof. We divide the proof into 5 cases:

Case 1. If $\max\{v, \hat{v}\} < -\hat{Q}_n$ and $\max\{w, \hat{w}\} < -\hat{Q}_n$ then it is easy to see that $\bar{\mathbf{F}}^0_{\hat{Q}_n}(v, \hat{v}) - \bar{\mathbf{F}}^0_{\hat{Q}_n}(w, \hat{w}) = 0$.

Case 2. If $\max\{v, \hat{v}\} < -\hat{Q}_n \leq \max\{w, \hat{w}\} \leq \hat{Q}_n$ then using triangle inequality, we get

$$\begin{aligned} \left| \bar{\mathbf{F}}^0_{\hat{Q}_n}(v, \hat{v}) - \bar{\mathbf{F}}^0_{\hat{Q}_n}(w, \hat{w}) \right| &= \left| \hat{Q}_n^2 - w\hat{w} \right| = \left| \hat{Q}_n (\hat{Q}_n + w) - w (\hat{Q}_n + \hat{w}) \right| \\ &\leq \hat{Q}_n |w + \hat{Q}_n| + |w| |\hat{w} + \hat{Q}_n| \\ &\leq \hat{Q}_n (|w + \hat{Q}_n| + |\hat{w} + \hat{Q}_n|) \leq \hat{Q}_n (|w - v| + |\hat{w} - \hat{v}|). \end{aligned}$$

Case 3. If $\max\{v, \hat{v}\} < -\hat{Q}_n < \hat{Q}_n \leq \max\{w, \hat{w}\}$ then

$$\left| \bar{\mathbf{F}}^0_{\hat{Q}_n}(v, \hat{v}) - \bar{\mathbf{F}}^0_{\hat{Q}_n}(w, \hat{w}) \right| = \left| \hat{Q}_n^2 - \hat{Q}_n^2 \right| = 0.$$

Case 4. If $-\hat{Q}_n < \max\{v, \hat{v}\}$, $\max\{w, \hat{w}\} \leq \hat{Q}_n$ then

$$\begin{aligned} \left| \bar{\mathbf{F}}^0_{\hat{Q}_n}(v, \hat{v}) - \bar{\mathbf{F}}^0_{\hat{Q}_n}(w, \hat{w}) \right| &= \left| v\hat{v} - w\hat{w} \right| = \left| (v - w)\hat{v} + w(\hat{v} - \hat{w}) \right| \\ &\leq |\hat{v}||v - w| + |w||\hat{v} - \hat{w}| \leq \hat{Q}_n(|v - \hat{v}| + |w - \hat{w}|). \end{aligned}$$

Case 5. If $\max\{v, \hat{v}\} > \hat{Q}_n$ and $\max\{w, \hat{w}\} > \hat{Q}_n$ then

$$\left| \bar{\mathbf{F}}^0_{\hat{Q}_n}(v, \hat{v}) - \bar{\mathbf{F}}^0_{\hat{Q}_n}(w, \hat{w}) \right| = \left| \hat{Q}_n^2 - \hat{Q}_n^2 \right| = 0.$$

By all cases above, we complete the proof of Lemma (2.2). \square

3 Regularized solutions for backward problem for Burgers' equation

Our main result in this paper is stated as follows

Theorem 3.1. *Let the functions $H \in \mathcal{H}^{\mu_0}(\Omega)$ and $A, G \in L^\infty(0, T; \mathcal{H}^{\mu_0}(\Omega))$, for $\mu_0 > \frac{1}{2}$. Then problem (2.13) has unique solution $\tilde{U}_{\rho_n} \in C([0, T]; L^2(\Omega))$. Assume that Problem (1.1) has unique solution $\mathbf{u} \in L^\infty(0, T; \mathcal{Z}_{\gamma, TA_1}(\Omega))$. Let us choose \hat{Q}_n such that*

$$\lim_{n \rightarrow +\infty} \exp\left(\frac{16|\hat{Q}_n|^2 T}{A_1 - A_0}\right) \max\left(e^{2\rho_n T} \beta_n^{1/2} n^{-4\mu}, e^{2\rho_n T} \beta_n^{-\mu_0}, \rho_n^{-2\gamma}\right) = 0. \quad (3.17)$$

Then for n large enough, $\mathbf{E} \|\tilde{U}_{\rho_n, \beta_n}(x, t) - \mathbf{u}(x, t)\|_{L^2(\Omega)}^2$ is of order

$$\exp\left(\frac{16|\hat{Q}_n|^2 T}{A_1 - A_0}\right) e^{-2\kappa_n t} \max\left(e^{2\rho_n T} \beta_n^{1/2} n^{-4\mu}, e^{2\rho_n T} \beta_n^{-\mu_0}, \rho_n^{-2\gamma}\right). \quad (3.18)$$

Proof. Denote by

$$B(x, t) = A_1 - A(x, t), \quad \bar{B}_{\beta_n}(x, t) = A_1 - \bar{A}_{\beta_n}(x, t). \quad (3.19)$$

The first equation of Problem (1.1) can be written as

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + \left(\bar{B}_{\beta_n}(x, t) \frac{\partial \mathbf{u}}{\partial x} \right)_x &= \mathbf{u} \mathbf{u}_x + \left((\bar{B}_{\beta_n}(x, t) - B(x, t)) \frac{\partial \mathbf{u}}{\partial x} \right)_x \\ &\quad + A_1 \Delta \mathbf{u} + G(x, t) \end{aligned} \quad (3.20)$$

and the first equation of Problem (2.13) is rewritten as

$$\begin{aligned} \frac{\partial \tilde{U}_{\rho_n, \beta_n}}{\partial t} + \left(\bar{B}_{\beta_n}(x, t) \frac{\partial \tilde{U}_{\rho_n, \beta_n}}{\partial x} \right)_x &= \bar{\mathbf{F}}^0_{\hat{Q}_n} \left(\tilde{U}_{\rho_n, \beta_n}, \frac{\partial \tilde{U}_{\rho_n, \beta_n}}{\partial x} \right) + \left((\bar{B}_{\beta_n}(x, t) - B(x, t)) \frac{\partial \tilde{U}_{\rho_n, \beta_n}}{\partial x} \right)_x \\ &\quad + \mathbf{P}_n \tilde{U}_{\rho_n, \beta_n} + \bar{G}_{\rho_n}(x, t). \end{aligned} \quad (3.21)$$

For $\kappa_n > 0$, we put

$$\mathbf{Y}_{\rho_n, \beta_n}(x, t) = e^{\kappa_n(t-T)} \left[\tilde{U}_{\rho_n, \beta_n}(x, t) - \mathbf{u}(x, t) \right].$$

Then the last two equations, and a simple computation gives

$$\begin{aligned} & \frac{\partial \mathbf{Y}_{\rho_n, \beta_n}}{\partial t} + \left(\bar{B}_{\rho_n} \frac{\partial \mathbf{Y}_{\rho_n, \beta_n}}{\partial x} \right)_x - \kappa_n \mathbf{Y}_{\rho_n, \beta_n} \\ &= \mathbf{P}_n \mathbf{Y}_{\rho_n, \beta_n} - e^{\kappa_n(t-T)} (\mathbf{P}_{\rho_n} - \mathbf{P}) \mathbf{u} - e^{\kappa_n(t-T)} \left((\bar{B}_{\beta_n}(x, t) - B(x, t)) \frac{\partial \mathbf{u}}{\partial x} \right)_x \\ &+ e^{\kappa_n(t-T)} \left[\bar{\mathbf{F}}^0_{\hat{Q}_n} \left(\tilde{U}_{\rho_n, \beta_n}, \frac{\partial \tilde{U}_{\rho_n, \beta_n}}{\partial x} \right) - \mathbf{u} \mathbf{u}_x \right] + e^{\kappa_n(t-T)} [\bar{G}_{\beta_n}(x, t) - G(x, t)] \end{aligned}$$

and $\mathbf{Y}_{\rho_n, \beta_n}|_{\partial\Omega} = 0$, $\mathbf{Y}_{\rho_n, \beta_n}(x, T) = \bar{H}_{\beta_n}(x) - H(x)$.

By taking the inner product of the two sides of the last equality with $\mathbf{Y}_{\rho_n, \beta_n}$ and noting the equality

$$\int_{\Omega} \left(\bar{B}_{\beta_n} \frac{\partial \mathbf{Y}_{\rho_n, \beta_n}}{\partial x} \right)_x \mathbf{Y}_{\rho_n, \beta_n} dx = - \int_{\Omega} \bar{B}_{\beta_n}(x, t) \left| \frac{\partial \mathbf{Y}_{\rho_n, \beta_n}}{\partial x} \right|^2 dx,$$

one deduces that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\mathbf{Y}_{\rho_n, \beta_n}(\cdot, t)\|_{L^2(\Omega)}^2 - \int_{\Omega} \bar{B}_{\beta_n}(x, t) \left| \frac{\partial \mathbf{Y}_{\rho_n, \beta_n}}{\partial x} \right|^2 dx - \kappa_n \|\mathbf{Y}_{\rho_n, \beta_n}(\cdot, t)\|_{L^2(\Omega)}^2 \\ &= \underbrace{\left\langle \mathbf{P}_n \mathbf{Y}_{\rho_n, \beta_n}, \mathbf{Y}_{\rho_n, \beta_n} \right\rangle_{L^2(\Omega)}}_{=:\tilde{\mathcal{J}}_{12,n}} + \underbrace{\left\langle e^{\kappa_n(t-T)} (\mathbf{P}_{\rho_n} - \mathbf{P}) \mathbf{u}, \mathbf{Y}_{\rho_n, \beta_n} \right\rangle_{L^2(\Omega)}}_{=:\tilde{\mathcal{J}}_{13,n}} \\ &+ \underbrace{\left\langle -e^{\kappa_n(t-T)} \left((\bar{B}_{\beta_n}(x, t) - B(x, t)) \frac{\partial \mathbf{u}}{\partial x} \right)_x, \mathbf{Y}_{\rho_n, \beta_n} \right\rangle_{L^2(\Omega)}}_{=:\tilde{\mathcal{J}}_{14,n}} \\ &+ \underbrace{\left\langle e^{\kappa_n(t-T)} \left[\bar{\mathbf{F}}^0_{\hat{Q}_n} \left(\tilde{U}_{\rho_n, \beta_n}, \frac{\partial \tilde{U}_{\rho_n, \beta_n}}{\partial x} \right) - \mathbf{u} \mathbf{u}_x \right], \mathbf{Y}_{\rho_n, \beta_n} \right\rangle_{L^2(\Omega)}}_{=:\tilde{\mathcal{J}}_{15,n}} \\ &+ \underbrace{\left\langle e^{\kappa_n(t-T)} [\bar{G}_{\beta_n}(x, t) - G(x, t)], \mathbf{Y}_{\rho_n, \beta_n} \right\rangle_{L^2(\Omega)}}_{=:\tilde{\mathcal{J}}_{16,n}}. \end{aligned}$$

For $\tilde{\mathcal{J}}_{12,n}$, we have the following

$$|\tilde{\mathcal{J}}_{12,n}| \leq \|\mathbf{P}_n \mathbf{Y}_{\rho_n}\|_{L^2(\Omega)} \|\mathbf{Y}_{\rho_n}(\cdot, t)\|_{L^2(\Omega)} \leq \rho_n \|\mathbf{Y}_{\rho_n}(\cdot, t)\|_{L^2(\Omega)}^2, \quad (3.22)$$

where we used inequality (2.9). And for $\tilde{\mathcal{J}}_{13,n}$, using Cauchy-Schwartz and (2.10), we have the following upper bound

$$|\tilde{\mathcal{J}}_{13,n}| \leq \frac{1}{2} e^{2\kappa_n(t-T)} A_1^2 \rho_n^{-2\gamma} e^{-2T\rho_n} \|\mathbf{u}\|_{L^\infty(0,T;\mathcal{Z}_{\gamma,T A_1}(\Omega))}^2 + \frac{1}{2} \|\mathbf{Y}_{\rho_n, \beta_n}(\cdot, t)\|_{L^2(\Omega)}^2. \quad (3.23)$$

The Cauchy-Schwartz inequality leads to the following estimation

$$\begin{aligned}
|\tilde{\mathcal{J}}_{14,n}| &= \left| \left\langle -e^{\kappa_n(t-T)} \left((\bar{B}_{\beta_n}(x,t) - B(x,t)) \frac{\partial \mathbf{u}}{\partial x} \right)_x, \mathbf{Y}_{\rho_n,\beta_n} \right\rangle_{L^2(\Omega)} \right| \\
&= \left| \left\langle -e^{\kappa_n(t-T)} \left((\bar{B}_{\beta_n}(x,t) - B(x,t)) \frac{\partial \mathbf{u}}{\partial x} \right), \frac{\partial \mathbf{Y}_{\rho_n,\beta_n}}{\partial x} \right\rangle_{L^2(\Omega)} \right| \\
&\leq \frac{e^{2\kappa_n(t-T)}}{2(A_1 - A_0)} \|\bar{B}_{\rho_n}(\cdot, t) - B(\cdot, t)\|_{L^2(\Omega)}^2 \left\| \frac{\partial \mathbf{u}}{\partial x}(\cdot, t) \right\|_{L^2(\Omega)}^2 + \frac{A_1 - A_0}{2} \int_{\Omega} \left| \frac{\partial \mathbf{Y}_{\rho_n,\beta_n}}{\partial x} \right|^2 dx \\
&\leq \frac{\|\bar{B}_{\beta_n}(\cdot, t) - B(\cdot, t)\|_{L^2(\Omega)}^2}{2(A_1 - A_0)} \|\mathbf{u}(\cdot, t)\|_{H_0^1(\Omega)}^2 + \frac{A_1 - A_0}{2} \left\| \frac{\partial \mathbf{Y}_{\rho_n,\beta_n}}{\partial x} \right\|_{L^2(\Omega)}^2.
\end{aligned}$$

For $\tilde{\mathcal{J}}_{15,n}$, we note that $\bar{\mathbf{F}}^0_{\hat{Q}_n}(\mathbf{u}, \mathbf{u}_x) = \mathbf{u}\mathbf{u}_x$ and thanks to (2.16), we obtain

$$\begin{aligned}
\left\| \bar{\mathbf{F}}^0_{\hat{Q}_n} \left(\tilde{U}_{\rho_n,\beta_n}, \frac{\partial \tilde{U}_{\rho_n,\beta_n}}{\partial x} \right) - \mathbf{u}\mathbf{u}_x \right\|_{L^2(\Omega)} &= \left\| \bar{\mathbf{F}}^0_{\hat{Q}_n} \left(\tilde{U}_{\rho_n,\beta_n}, \frac{\partial \tilde{U}_{\rho_n,\beta_n}}{\partial x} \right) - \bar{\mathbf{F}}^0_{\hat{Q}_n}(\mathbf{u}, \mathbf{u}_x) \right\|_{L^2(\Omega)} \\
&\leq \hat{Q}_n \left(\left\| \tilde{U}_{\rho_n,\beta_n} - \mathbf{u} \right\|_{L^2(\Omega)} + \left\| \frac{\partial \tilde{U}_{\rho_n,\beta_n}}{\partial x} - \mathbf{u}_x \right\|_{L^2(\Omega)} \right) \\
&= e^{\kappa_n(T-t)} \hat{Q}_n \left(\left\| \mathbf{Y}_{\rho_n,\beta_n} \right\|_{L^2(\Omega)} + \left\| \frac{\partial \mathbf{Y}_{\rho_n,\beta_n}}{\partial x} \right\|_{L^2(\Omega)} \right) \\
&\leq 2e^{\kappa_n(T-t)} \hat{Q}_n \left\| \frac{\partial \mathbf{Y}_{\rho_n,\beta_n}}{\partial x} \right\|_{L^2(\Omega)},
\end{aligned}$$

where we note that $\|\mathbf{Y}_{\rho_n,\beta_n}\|_{L^2(\Omega)} \leq \left\| \frac{\partial \mathbf{Y}_{\rho_n,\beta_n}}{\partial x} \right\|_{L^2(\Omega)}$. This implies that

$$\begin{aligned}
|\tilde{\mathcal{J}}_{15,n}| &= \left| \left\langle e^{\kappa_n(t-T)} \left[\bar{\mathbf{F}}^0_{\hat{Q}_n} \left(\tilde{U}_{\rho_n,\beta_n}, \frac{\partial \tilde{U}_{\rho_n,\beta_n}}{\partial x} \right) - \mathbf{u}\mathbf{u}_x \right], \mathbf{Y}_{\rho_n,\beta_n} \right\rangle_{L^2(\Omega)} \right| \\
&\leq e^{2\kappa_n(t-T)} \frac{A_1 - A_0}{8|\hat{Q}_n|^2} \left\| \bar{\mathbf{F}}^0_{\hat{Q}_n} \left(\tilde{U}_{\rho_n,\beta_n}, \frac{\partial \tilde{U}_{\rho_n,\beta_n}}{\partial x} \right) - \mathbf{u}\mathbf{u}_x \right\|_{L^2(\Omega)}^2 + \frac{8|\hat{Q}_n|^2}{A_1 - A_0} \|\mathbf{Y}_{\rho_n,\beta_n}(\cdot, t)\|_{L^2(\Omega)}^2 \\
&\leq \frac{A_1 - A_0}{2} \left\| \frac{\partial \mathbf{Y}_{\rho_n,\beta_n}}{\partial x} \right\|_{L^2(\Omega)}^2 + \frac{8|\hat{Q}_n|^2}{A_1 - A_0} \|\mathbf{Y}_{\rho_n,\beta_n}(\cdot, t)\|_{L^2(\Omega)}^2.
\end{aligned}$$

The term $|\tilde{\mathcal{J}}_{16,n}|$ can be bounded by

$$\begin{aligned}
|\tilde{\mathcal{J}}_{16,n}| &= \left| \left\langle e^{\kappa_n(t-T)} [\bar{G}_{\beta_n}(\cdot, t) - G(\cdot, t)], \mathbf{Y}_{\rho_n,\beta_n} \right\rangle_{L^2(\Omega)} \right| \\
&\leq \frac{1}{2} e^{2\kappa_n(t-T)} \|\bar{G}_{\beta_n}(\cdot, t) - G(\cdot, t)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\mathbf{Y}_{\rho_n,\beta_n}\|_{L^2(\Omega)}^2.
\end{aligned}$$

Combining all the previous estimates, we get

$$\begin{aligned}
& \frac{d}{dt} \|\mathbf{Y}_{\rho_n, \beta_n}(\cdot, t)\|_{L^2(\Omega)}^2 - 2 \int_{\Omega} \overline{B}_{\beta_n}(x, t) \left| \frac{\partial \mathbf{Y}_{\rho_n, \beta_n}}{\partial x} \right|^2 dx - 2\kappa_n \|\mathbf{Y}_{\rho_n, \beta_n}(\cdot, t)\|_{L^2(\Omega)}^2 \\
& \geq -2\rho_n \|\mathbf{Y}_{\rho_n, \beta_n}(\cdot, t)\|_{L^2(\Omega)}^2 - e^{2\kappa_n(t-T)} A_1^2 \rho_n^{-2\gamma} e^{-2T\rho_n} \|\mathbf{u}\|_{L^\infty(0, T; \mathcal{Z}_{\gamma, TA_1}(\Omega))}^2 \\
& \quad - \|\mathbf{Y}_{\rho_n, \beta_n}(\cdot, t)\|_{L^2(\Omega)}^2 - \frac{\|\overline{B}_{\beta_n}(\cdot, t) - B(\cdot, t)\|_{L^2(\Omega)}^2}{(A_1 - A_0)} \|\mathbf{u}(\cdot, t)\|_{H_0^1(\Omega)}^2 \\
& \quad - 2(A_1 - A_0) \left\| \frac{\partial \mathbf{Y}_{\rho_n, \beta_n}}{\partial x} \right\|_{L^2(\Omega)}^2 - \frac{16|\widehat{Q}_n|^2}{A_1 - A_0} \|\mathbf{Y}_{\rho_n, \beta_n}(\cdot, t)\|_{L^2(\Omega)}^2 \\
& \quad - e^{2\kappa_n(t-T)} \|\overline{G}_{\beta_n}(\cdot, t) - G(\cdot, t)\|_{L^2(\Omega)}^2 - \|\mathbf{Y}_{\rho_n, \beta_n}\|_{L^2(\Omega)}^2.
\end{aligned}$$

By taking the integral from t to T and by a simple calculation yields

$$\begin{aligned}
& \|\mathbf{Y}_{\rho_n, \beta_n}(\cdot, T)\|_{L^2(\Omega)}^2 - \|\mathbf{Y}_{\rho_n, \beta_n}(\cdot, t)\|_{L^2(\Omega)}^2 \\
& + \int_t^T \left(A_1^2 \rho_n^{-2\gamma} e^{-2T\rho_n} \|\mathbf{u}\|_{L^\infty(0, T; \mathcal{Z}_{\gamma, TA_1}(\Omega))}^2 + \frac{\|\overline{B}_{\beta_n}(x, t) - B(x, t)\|_{L^2(\Omega)}^2}{(A_1 - A_0)} \|\mathbf{u}\|_{L^\infty((0, T); H_0^1(\Omega))}^2 \right) ds \\
& \geq 2 \int_t^T \int_{\Omega} \left(\overline{B}_{\rho_n}(x, s) - (A_1 - A_0) \right) \left\| \frac{\partial \mathbf{Y}_{\rho_n, \beta_n}}{\partial x} \right\|_{L^2(\Omega)}^2 dx ds \\
& \quad + \int_t^T \left(2\kappa_n - 2\rho_n - \frac{16|\widehat{Q}_n|^2}{A_1 - A_0} - 2 \right) \|\mathbf{Y}_{\rho_n, \beta_n}(\cdot, s)\|_{L^2(\Omega)}^2 ds \\
& \quad - T e^{2\kappa_n(t-T)} \|\overline{G}_{\beta_n}(\cdot, t) - G(\cdot, t)\|_{L^\infty(0, T; L^2(\Omega))}^2 \\
& \geq \int_t^T \left(2\kappa_n - 2\rho_n - \frac{16|\widehat{Q}_n|^2}{A_1 - A_0} - 2 \right) \|\mathbf{Y}_{\rho_n, \beta_n}(\cdot, s)\|_{L^2(\Omega)}^2 ds \\
& \quad - T e^{2\kappa_n(t-T)} \|\overline{G}_{\beta_n}(\cdot, t) - G(\cdot, t)\|_{L^\infty(0, T; L^2(\Omega))}^2.
\end{aligned}$$

where we used the fact that

$$\overline{B}_{\beta_n}(x, s) = A_1 - \overline{A}_{\rho_n}(x, s) \geq A_1 - A_0.$$

Let us choose $\kappa_n = \rho_n$ then we obtain

$$\begin{aligned}
& e^{2\kappa_n(t-T)} \mathbf{E} \|\tilde{U}_{\rho_n, \beta_n}(\cdot, t) - \mathbf{u}(\cdot, t)\|_{L^2(\Omega)}^2 \\
& \leq \mathbf{E} \|\overline{H}_{\beta_n} - H\|_{L^2(\Omega)}^2 + T A_1^2 \rho_n^{-2\gamma} e^{-2T\rho_n} \|\mathbf{u}\|_{L^\infty(0, T; \mathcal{Z}_{\gamma, TA_1}(\Omega))}^2 \\
& \quad + T \mathbf{E} \|\overline{G}_{\beta_n}(\cdot, t) - G(\cdot, t)\|_{L^\infty(0, T; L^2(\Omega))}^2 \\
& \quad + \frac{\mathbf{E} \|\overline{B}_{\beta_n}(\cdot, t) - B(\cdot, t)\|_{L^\infty(0, T; L^2(\Omega))}^2}{(A_1 - A_0)} \|\mathbf{u}\|_{L^\infty((0, T); H_0^1(\Omega))}^2 \\
& \quad + \left(\frac{16|\widehat{Q}_n|^2}{A_1 - A_0} + 2 \right) \int_t^T e^{2\kappa_n(s-T)} \mathbf{E} \|\tilde{U}_{\rho_n, \beta_n}(\cdot, s) - \mathbf{u}(\cdot, s)\|_{L^2(\Omega)}^2 ds.
\end{aligned}$$

Multiplying both sides of the last inequality by $e^{2\kappa_n T}$, we obtain

$$\begin{aligned}
& e^{2\kappa_n t} \mathbf{E} \|\tilde{U}_{\rho_n, \beta_n}(\cdot, t) - \mathbf{u}(\cdot, t)\|_{L^2(\Omega)}^2 \\
& \leq e^{2\kappa_n T} \mathbf{E} \|\bar{H}_{\beta_n} - H\|_{L^2(\Omega)}^2 + T A_1^2 \rho_n^{-2\gamma} \|\mathbf{u}\|_{L^\infty(0, T; \mathcal{Z}_{\gamma, T A_1}(\Omega))}^2 \\
& + T e^{2\kappa_n T} \mathbf{E} \|\bar{G}_{\beta_n} - G\|_{L^\infty(0, T; L^2(\Omega))}^2 \\
& + e^{2\kappa_n T} \frac{\mathbf{E} \|\bar{B}_{\beta_n} - B\|_{L^\infty(0, T; L^2(\Omega))}^2}{(A_1 - A_0)} \|\mathbf{u}\|_{L^\infty((0, T); H_0^1(\Omega))}^2 \\
& + \left(\frac{16|\hat{Q}_n|^2}{A_1 - A_0} + 2 \right) \int_t^T e^{2s\kappa_n} \mathbf{E} \|\tilde{U}_{\rho_n, \beta_n}(\cdot, s) - \mathbf{u}(\cdot, s)\|_{L^2(\Omega)}^2 ds.
\end{aligned}$$

Applying Gronwall's inequality, we deduce that

$$\mathbf{E} \|\tilde{U}_{\rho_n, \beta_n}(x, t) - \mathbf{u}(x, t)\|_{L^2(\Omega)}^2 \leq \exp \left(\frac{16|\hat{Q}_n|^2(T-t)}{A_1 - A_0} + 2(T-t) \right) e^{-2\kappa_n t} B' \quad (3.24)$$

where

$$\begin{aligned}
B' &= e^{2\kappa_n T} \mathbf{E} \|\bar{H}_{\beta_n} - H\|_{L^2(\Omega)}^2 + T A_1^2 \rho_n^{-2\gamma} \|\mathbf{u}\|_{L^\infty(0, T; \mathcal{Z}_{\gamma, T A_1}(\Omega))}^2 \\
&+ T e^{2\kappa_n T} \mathbf{E} \|\bar{G}_{\beta_n} - G\|_{L^\infty(0, T; L^2(\Omega))}^2 + e^{2\kappa_n T} \frac{\mathbf{E} \|\bar{B}_{\beta_n} - B\|_{L^\infty(0, T; L^2(\Omega))}^2}{(A_1 - A_0)} \|\mathbf{u}\|_{L^\infty((0, T); H_0^1(\Omega))}^2.
\end{aligned}$$

Thanks to Theorem 2.1, we have that

$$\mathbf{E} \left\| \hat{H}_{\beta_n} - H \right\|_{L^2(\Omega)}^2 + T \mathbf{E} \left\| \hat{G}_{\beta_n} - G \right\|_{L^\infty(0, T; L^2(\Omega))}^2 + \mathbf{E} \|\bar{B}_{\beta_n} - B\|_{L^\infty(0, T; L^2(\Omega))}^2$$

is of order $\max \left(\beta_n^{1/2} n^{-4\mu}, \beta_n^{-\mu_0} \right)$ for any $\mu > \frac{1}{2}$. This together with (3.24) implies that $\mathbf{E} \|\tilde{U}_{\rho_n, \beta_n}(\cdot, t) - \mathbf{u}(\cdot, t)\|_{L^2(\Omega)}^2$ is of order

$$\exp \left(\frac{16|\hat{Q}_n|^2 T}{A_1 - A_0} \right) e^{-2\kappa_n t} \max \left(e^{2\rho_n T} \beta_n^{1/2} n^{-4\mu}, e^{2\rho_n T} \beta_n^{-\mu_0}, \rho_n^{-2\gamma} \right). \quad (3.25)$$

□

References

- [1] A. Carasso, *Computing small solutions of Burgers' equation backwards in time* J. Math. Anal. Appl. 59 (1977), no. 1, 169-209.
- [2] R.L. Eubank. *Nonparametric regression and spline smoothing*. Second edition. Statistics: Textbooks and Monographs, 157. Marcel Dekker, Inc., New York, 1999. xii+338 pp.
- [3] D.N. Hao, N.V. Duc, N.V. Thang, *Stability estimates for Burgers'-type equations backward in time* J. Inverse Ill-Posed Probl. 23 (2015), no. 1, 41-49.

- [4] M. Kirane, E. Nane and N. H. Tuan *On a backward problem for multidimensional Ginzburg-Landau equation with random data* Submitted, 2017. URL: <https://arxiv.org/abs/1702.03024>
- [5] M. Kirane, E. Nane and N. H. Tuan *Regularized solutions for some backward nonlinear partial differential equations with statistical data* Submitted, 2017. URL: <https://arxiv.org/abs/1701.08459>
- [6] A. Kirsch *An introduction to the mathematical theory of inverse problems. Second edition* Applied Mathematical Sciences, 120. Springer, New York, 2011. xiv+307 pp.
- [7] I. Kukavica, *Log-log convexity and backward uniqueness*, Proc. Amer. Math. Soc. 135 (2007), 2415-2421
- [8] D.D. Trong, T.D. Khanh, N.H. Tuan, N.D. Minh, *Nonparametric regression in a statistical modified Helmholtz equation using the Fourier spectral regularization* Statistics 49 (2015), no. 2, 267-290.
- [9] N. H. Tuan and E. Nane. *Inverse source problem for time fractional diffusion with discrete random noise*. Statistics and Probability Letters. Volume 120, January 2017, Pages 126-134.